

# A NOTE ON THE CONE RESTRICTION CONJECTURE IN THE CYLINDRICALLY SYMMETRIC CASE

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ABSTRACT. In this note, we present two arguments showing that the classical *linear adjoint cone restriction conjecture* holds for the class of functions supported on the cone and invariant under the spatial rotation in all dimensions. The first is based on a dyadic restriction estimate, while the second follows from a strengthening version of the Hausdorff-Young inequality and the Hölder inequality in the Lorentz spaces.

## 1. INTRODUCTION

Let  $n \geq 2$  be a fixed integer and  $S$  be a smooth compact non-empty subset of the cone  $\{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^n : \tau = |\xi|\}$ , where we interpret  $\mathbf{R} \times \mathbf{R}^n$  as the time-space frequency space. If  $0 < p, q \leq \infty$ , the classical *linear adjoint restriction estimate*<sup>1</sup> for the cone is the following “*a priori*” estimate

$$(1.1) \quad \|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbf{R} \times \mathbf{R}^n)} \leq C_{p,q,n,S} \|f\|_{L^p(S,d\sigma)}$$

for all Schwartz functions  $f$  on  $S$ , where

$$(fd\sigma)^\vee(t, x) = \int_S f(\tau, \xi) e^{i(x \cdot \xi + t\tau)} d\sigma(\xi) = \int_{\mathbf{R}^n} f(|\xi|, \xi) e^{i(x \cdot \xi + t|\xi|)} \frac{d\xi}{|\xi|}$$

denotes the inverse space-time Fourier transform of the measure  $fd\sigma$ , and  $d\sigma$  is the pull-back of the measure  $\frac{d\xi}{|\xi|}$  under the projection map  $(\tau, \xi) \mapsto \xi$ . By duality, the estimate (1.1) is equivalent to

$$\|\hat{f}\|_{L^{p'}(S,d\sigma)} \leq C_{p,q,n,S} \|f\|_{L^{q'}(\mathbf{R} \times \mathbf{R}^n)}$$

for all Schwartz functions  $f$ , which roughly says that the Fourier transform of an  $L^{q'}(\mathbf{R} \times \mathbf{R}^n)$  function can be “meaningfully” restricted to the cone  $S$ . This leads to the *restriction problem*, one of the central problems in harmonic analysis, which concerns the optimal range of exponents  $p$  and  $q$  for which the estimate (1.1) should hold. It was originally proposed by Stein for the sphere [5] and then extended to smooth sub-manifolds of  $\mathbf{R} \times \mathbf{R}^n$  with appropriate curvature [6, Chapter 8, pages 352-355, 364-367] such as the paraboloid and the cone. The restriction problem is intricately related to other outstanding problems in analysis such as the Bochner-Riesz conjecture, the Sogge’s local smoothing conjecture, the Kakeya set conjecture and the Kakeya maximal function conjecture, see e.g., [9], [10].

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<sup>1</sup>In the notation of [9], the estimate (1.1) is denoted by  $R^*_S(p \rightarrow q)$ .

By testing (1.1) against the characteristic functions supported on a symmetric band or a small cap of the cone, the following conjecture on the restriction of the Fourier transform to the cone can be formulated,

**Conjecture 1.1** (Linear adjoint cone restriction conjecture). *The inequality (1.1) holds with constants depending on  $S$ ,  $n$  and  $p, q$  if and only if  $q > \frac{2n}{n-1}$  and  $\frac{n+1}{q} \leq \frac{n-1}{p'}$ .*

Córdoba and Stein proved that (1.1) was true under the condition  $p = 2$  and  $q \geq \frac{2(n+1)}{n-1}$  in an unpublished work. Strichartz [8] then extended the results to more general quadratic surfaces. In 1985, Barcelo [1] proved Conjecture 1.1 when  $n = 2$ . A major breakthrough was made in 2001 by Wolff [14], who showed that Conjecture 1.1 was true when  $n = 3$ . This was based on a new bilinear cone restriction estimate, which also gave the current best result  $q > \frac{2(n+3)}{n+1}$  in higher dimensions  $n \geq 4$ . We should remark that all the recent progress on the linear restriction is achieved from the corresponding bilinear restriction estimates, especially the bilinear  $L^2$ -type estimates,  $L^2 \times L^2 \rightarrow L^q$  for some  $q \in [1, 2]$ ; more information about the so-called bilinear method and recent ideas of attacking the restriction conjecture such as the reduction to the local restriction estimates, the wave packet decomposition and the induction-on-scales can be found in [13], [11], [14], [12] and [9].

When we restrict the functions supported on the cone  $S$  to be cylindrically symmetric, i.e., functions invariant under the spatial rotation, the following theorem is our main result in this paper,

**Theorem 1.1.** *Conjecture 1.1 holds for cylindrically symmetric functions supported on the cone in all dimensions.*

For the same class of functions but supported on the paraboloid, the author [4] has verified the corresponding conjecture for the paraboloid in all dimensions. The first proof of Theorem 1.1 is along similar lines as in [4], through dyadically decomposing both the frequency and spatial spaces and then establishing a family of dyadical restriction estimates based on the “Fourier-Bessel” formula defined in Section 3. The second proof is inspired by Nicola’s argument on the implication of cone restriction conjecture from the sphere restriction conjecture in [3]. The key ingredient is the use of the strengthening version of the Hausdorff-Young inequality [7, Chapter 4, Corollary 3.16] and the Hölder inequality in the Lorentz spaces [2, Chapter 5, Theorem 5.3.1].

*Remark 1.2.* As in [4], for the cylindrically symmetric functions with dydical supports, we expect that more estimates are available. This is indeed the case: from Proposition 3.1 and Corollary 3.4, when  $f$  is cylindrically symmetric and supported on a subset of the cone  $\{(|\xi|, \xi) : 1 \leq |\xi| \leq 2\}$ , for  $q > \frac{2n}{n-1}$  and  $q \geq p'$ ,

$$\|(f d\sigma)^\vee\|_{L^q(\mathbf{R} \times \mathbf{R}^n)} \leq C_{p,q} \|f\|_{L^p(S, d\sigma)}.$$

We note that  $q \geq p'$  is an improvement over  $q \geq \frac{n+1}{n-1}p'$ .

*Remark 1.3.* When  $S$  is the whole cone instead of a compact subset of the cone, we see that the necessary conditions are strengthened to

$$q > \frac{2n}{n-1}, \quad \frac{n+1}{q} = \frac{n-1}{p'}.$$

In this case, on the one hand, Theorem 1.1 guarantees that the cone restriction conjecture 1.1 is true; on the other hand, unlike the situation in Remark 1.2, there are no more estimates available.

This paper is organized as follows. Section 2 is devoted to establishing the standard notations; in Section 3 we present our first proof of Theorem 1.1 via the dyadic restriction estimates; in Section 4 we present another proof by using a strengthening version of the Hausdorff-Young inequality and the Hölder inequality in the Lorentz spaces.

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## 2. NOTATIONS

We will use the notations  $X \lesssim Y$ ,  $Y \gtrsim X$ , or  $X = O(Y)$  to denote the estimate  $|X| \leq CY$  for some constant  $0 < C < \infty$ , which may depend on  $p, q, n$  and  $S$ , but not on the functions. If  $X \lesssim Y$  and  $Y \lesssim X$  we will write  $X \sim Y$ . If the constant  $C$  depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For example,  $C_\varepsilon$  should be understood as a positive constant not only depending on  $p, q, n$  and  $S$ , but also on  $\varepsilon$ .

By  $\mathcal{S}^{n-1}$  we denote the  $n - 1$  dimensional unit sphere, and by  $d\mu$  the canonical surface measure of the sphere. We define a dyadic number to be any number  $R \in 2^{\mathbb{Z}}$  of the form  $R = 2^j$  where  $j$  is an integer. For each dyadic number  $R > 0$ , we define the dyadic annulus in  $\mathbf{R}^n$ ,  $A_R := \{x \in \mathbf{R}^n : R/2 \leq |x| \leq R\}$ . By  $L_N$ , we denote the class of cylindrically symmetric functions dyadically supported on the cone, i.e., functions invariant under the spatial rotation and supported on a set of the form  $\{(\tau, \xi) : N \leq |\xi| \leq 2N, \tau = |\xi|\}$  with dyadic  $N > 0$ . We define the spacetime norm  $L_t^q L_x^r$  of  $f$  on  $\mathbf{R} \times \mathbf{R}^n$  by

$$\|f\|_{L_t^q L_x^r(\mathbf{R} \times \mathbf{R}^n)} := \left( \int_{\mathbf{R}} \left( \int_{\mathbf{R}^n} |f(t, x)|^r dx \right)^{q/r} dt \right)^{1/q}$$

with the usual modifications when  $q$  or  $r$  are equal to infinity, or when the domain  $\mathbf{R} \times \mathbf{R}^n$  is replaced by a small region of spacetime such as  $\mathbf{R} \times A_R$ . When  $q = r$ , we abbreviate it by  $L_{t,x}^q$ . We define the spatial Fourier transform of  $f$  on  $\mathbf{R}^n$  by  $\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-ix \cdot \xi} dx$ . We use  $1_U$  to denote the characteristic function of the set  $U$ , i.e.,  $1_U(x) := 1$  if  $x \in U$ , otherwise 0. For  $1 \leq p \leq \infty$ , we denote the conjugate exponent of  $p$  by  $p'$ , i.e.,  $1/p + 1/p' = 1$ .

## 3. FIRST PROOF OF THEOREM 1.1

For any cylindrically symmetric function  $f$  on the cone, we set  $F(|\xi|) := f(|\xi|, \xi)$ . We observe that  $(fd\sigma)^\vee(t, x)$  is also a cylindrically symmetric function. To begin the proof of Theorem 1.1, we investigate the behavior of  $(fd\sigma)^\vee$  on  $\{|x| \leq 1\}$  via the following proposition.

**Proposition 3.1.** *Suppose  $f \in L_1$ . Then for any  $1 \leq p \leq \infty$ ,  $q \geq \max\{2, p'\}$  and  $R \leq 1$ , we have*

$$(3.1) \quad \|(fd\sigma)^\vee\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim R^{\frac{n}{q}} \|f\|_{L^p(S, d\sigma)}.$$

*Proof.* If we change to polar coordinates, the left-hand side of (3.1) is

$$\begin{aligned}
& \left( \int_{A_R} \int_{\mathbf{R}} \left| \int_{1 \leq |\xi| \leq 2} f(|\xi|, \xi) e^{i(x \cdot \xi + t|\xi|)} \frac{d\xi}{|\xi|} \right|^q dt dx \right)^{1/q} \\
&= \left( \int_{R/2}^R \int_{\mathbf{R}} \left| \int_{1 \leq |\xi| \leq 2} f(|\xi|, \xi) e^{i(re_1 \cdot \xi + t|\xi|)} \frac{d\xi}{|\xi|} \right|^q dt r^{n-1} dr \right)^{1/q} \\
&= \left( \int_{R/2}^R \int_{\mathbf{R}} \left| \int_I F(s) s^{n-2} e^{its} \int_{S^{n-1}} e^{irse_1 \cdot \omega} d\mu(\omega) ds \right|^q dt r^{n-1} dr \right)^{1/q} \\
&= \left( \int_{R/2}^R \int_{\mathbf{R}} \left| \int_I F(s) s^{n-2} e^{its} (d\mu)^\vee(rse_1 \cdot \omega) ds \right|^q dt r^{n-1} dr \right)^{1/q},
\end{aligned}$$

where  $I = [1, 2]$ ,  $e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n$  and “ $\cdot$ ” denotes the inner product operation in  $\mathbf{R}^n$ . Then from the Hausdorff-Young inequality when  $q > 2$  or Plancherel theorem when  $q = 2$  and using  $\|(d\mu)^\vee\|_{L^\infty} \lesssim 1$ , the left-hand side of (3.1) is further bounded by

$$R^{\frac{n-1}{q}} \left( \int_{R/2}^R \|F\|_{L^{q'}(I)}^q dr \right)^{1/q} \sim R^{\frac{n}{q}} \|F\|_{L^{q'}(I)}.$$

Then by applying the Hölder inequality to raising  $q'$  to  $p$  since  $p \geq q'$ , and noting  $\|F\|_{L^p(I)} \sim \|f\|_{L^p(S, d\sigma)}$ , (3.1) follows.  $\square$

Before investigating the behavior of  $(fd\sigma)^\vee$  on  $|x| \geq 1$ , we exploit the cylindrical symmetry of  $f$  in the following proposition. Note that we will encode the error term of the Bessel function into integrals instead of using its asymptotic bound.

**Lemma 3.2** (Fourier-Bessel formula). *Suppose  $f$  is a cylindrically symmetric function supported on the cone. Then*

$$\begin{aligned}
& (fd\sigma)^\vee(t, x) \\
&= c_n r^{-\frac{n-1}{2}} \int_I F(s) s^{\frac{n-3}{2}} e^{i(rs+ts)} ds + c_n r^{-\frac{n-1}{2}} \int_I F(s) s^{\frac{n-3}{2}} e^{i(-rs+ts)} ds \\
&\quad + c_n \int_I F(s) s^{n-2} e^{its-irs} \int_0^\infty e^{-rsy} y^{\frac{n-3}{2}} [(y+2i)^{\frac{n-3}{2}} - (2i)^{\frac{n-3}{2}}] dy ds \\
&\quad + c_n \int_I F(s) s^{n-2} e^{its+irs} \int_0^\infty e^{-rsy} y^{\frac{n-3}{2}} [(y-2i)^{\frac{n-3}{2}} - (-2i)^{\frac{n-3}{2}}] dy ds.
\end{aligned}$$

where  $I$  denotes the interval in the radial direction and  $r = |x|$ .

*Proof.* We first expand  $(fd\sigma)^\vee$  in the polar coordinates,

$$(fd\sigma)^\vee(t, x) = \int_{\{|\xi| \in I\}} f(|\xi|, \xi) e^{i(re_1 \cdot \xi + t|\xi|)} \frac{d\xi}{|\xi|} = \int_I F(s) e^{its} s^{n-2} (d\mu)^\vee(rse_1 \cdot \omega) ds.$$

We recall  $(d\mu)^\vee(\xi) = c_n |\xi|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(|\xi|)$ , see e.g., [6, page 347]. Moreover from [7, Chapter 3, Lemma 11], we obtain that, for fixed  $m \geq 0$ ,

$$\begin{aligned} J_m(r) &= c_m r^{-1/2} (e^{ir} - e^{-ir}) \\ &\quad + c_m r^m e^{-ir} \int_0^\infty e^{-ry} y^{\frac{2m-1}{2}} [(y+2i)^{\frac{2m-1}{2}} - (2i)^{\frac{2m-1}{2}}] dy \\ &\quad + c_m r^m e^{ir} \int_0^\infty e^{-ry} y^{\frac{2m-1}{2}} [(y-2i)^{\frac{2m-1}{2}} - (-2i)^{\frac{2m-1}{2}}] dy. \end{aligned}$$

Then Lemma 3.2 follows after we combine these two estimates and set  $m = \frac{n-2}{2}$ .  $\square$

In view of the previous lemma, we thus define the main term and the error term of  $(fd\sigma)^\vee$  by

$$\begin{aligned} \mathcal{M}f(t, x) &:= c_n r^{-\frac{n-1}{2}} \int_I F(s) s^{\frac{n-3}{2}} e^{i(rs+ts)} ds + c_n r^{-\frac{n-1}{2}} \int_I F(s) s^{\frac{n-3}{2}} e^{i(-rs+ts)} ds, \\ \mathcal{E}f(t, x) &:= c_n \int_I F(s) s^{n-2} e^{its+irs} \int_0^\infty e^{-rsy} y^{\frac{n-3}{2}} [(y+2i)^{\frac{n-3}{2}} - (2i)^{\frac{n-3}{2}}] dy ds \\ &\quad - c_n \int_I F(s) s^{n-2} e^{its-irs} \int_0^\infty e^{-rsy} y^{\frac{n-3}{2}} [(y-2i)^{\frac{n-3}{2}} - (-2i)^{\frac{n-3}{2}}] dy ds. \end{aligned}$$

Heuristically, one should think of  $\mathcal{E}f$  as  $r^{-(n+1)/2} \int_I F(s) s^{\frac{n-5}{2}} e^{its} ds$ , which is given by estimating the error term of Bessel function  $J_m(r)$  by  $r^{-3/2}$ . The following proposition shows that the error term estimate is acceptable compared to the main term estimate.

**Proposition 3.3.** *Suppose  $f \in L_1$ . Then for all  $1 \leq p \leq \infty$ ,  $q \geq \max\{2, p'\}$ , a dyadic number  $R \geq 2$  and  $f \in L^p(S, d\sigma)$ , we have the main term estimate,*

$$(3.2) \quad \|\mathcal{M}f\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim R^{-\frac{n-1}{2}[1-\frac{2n}{q(n-1)}]} \|f\|_{L^p(S, d\sigma)},$$

and the error term estimate

$$(3.3) \quad \|\mathcal{E}f\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim R^{-\frac{n+1}{2}+\frac{n}{q}} \|f\|_{L^p(S, d\sigma)}.$$

*Proof.* To prove the main term estimate (3.2), we first observe that it is sufficient to obtain the same estimate with the first term in the expression of  $\mathcal{M}f$ . Then by changing to polar coordinates and the Hausdorff-Young inequality in  $t$  when  $q > 2$  or the Plancherel theorem in  $t$  when  $q = 2$ , we obtain

$$\begin{aligned} \|\mathcal{M}f\|_{L_{t,x}^q(\mathbf{R} \times A_R)} &\sim \left( \int_{R/2}^R \int_{\mathbf{R}} \left| r^{-\frac{n-1}{2}} \int_I F(s) s^{\frac{n-3}{2}} e^{i(rs+ts)} ds \right|^q dt r^{n-1} dr \right)^{1/q} \\ &= R^{-\frac{n-1}{2}+\frac{n-1}{q}} \left( \int_{R/2}^R \int_{\mathbf{R}} \left| \int_I F(s) s^{\frac{n-3}{2}} e^{irs} e^{its} ds \right|^q dt dr \right)^{1/q} \\ &\lesssim R^{-\frac{n-1}{2}+\frac{n-1}{q}} \left( \int_{R/2}^R \|F\|_{L_s^{q'}(I)}^q dr \right)^{1/q} \lesssim R^{-\frac{n-1}{2}+\frac{n}{q}} \|f\|_{L^p(S, d\sigma)}. \end{aligned}$$

Hence (3.2) follows.

To prove the error term estimate (3.3), for  $r \geq 1$ , we set

$$E(r) = \int_0^\infty e^{-ry} y^{\frac{n-3}{2}} [(y \pm 2i)^{\frac{n-3}{2}} - (\pm 2i)^{\frac{n-3}{2}}] dy.$$

By a similar argument as proving [7, Chapter 3, Lemma 3.11] (details can also be found in [4, Proposition 3.3]), we have

$$(3.4) \quad |E(r)| \lesssim r^{-\frac{n+1}{2}}, \text{ for } r \geq 1.$$

By changing to polar coordinates, the left-hand side of (3.3) is comparable to

$$\left( \int_{R/2}^R \int_{\mathbf{R}} \left| \int_I F(s) s^{n-2} e^{its \pm irs} E(rs) ds \right|^q dt r^{n-1} dr \right)^{1/q}.$$

Then by the Hausdorff-Young inequality in  $t$  when  $q > 2$  or Plancherel theorem in  $t$  when  $q = 2$  and  $s \sim 1$ , it is further bounded by

$$\left( \int_{R/2}^R \left| \int_I |F(s) s^{n-2} E(rs)|^{q'} ds \right|^{q/q'} r^{n-1} dr \right)^{1/q}$$

By using (3.4) and Hölder since  $q \geq p'$ , it is bounded by  $R^{-\frac{n+1}{2} + \frac{n}{q}} \|F\|_{L^p(I)}$ . Then (3.3) follows because  $\|F\|_{L^p(I)} \sim \|f\|_{L^p(S, d\sigma)}$ .  $\square$

From the triangle inequality, we have

**Corollary 3.4** (Dyadic restriction estimate). *Suppose  $f \in L_1$ . Then for all  $1 \leq p \leq \infty$ ,  $q \geq \max\{2, p'\}$ , a dyadic number  $R \geq 2$  and  $f \in L^p(S, d\sigma)$ , we have*

$$\|(fd\sigma)^\vee\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim R^{-\frac{n-1}{2} [1 - \frac{2n}{q(n-1)}]} \|f\|_{L^p(S, d\sigma)}.$$

Having done all the preparations, we now prove Theorem 1.1 via the dyadic restriction estimate above.

*The first proof of Theorem 1.1.* We only need to show the “sufficient” part of the claim. We first observe that it suffices to prove (1.1) under the boundary condition  $q > \frac{2n}{n-1}$  and  $\frac{n+1}{q} = \frac{n-1}{p'}$  since other estimates are easily obtained by a standard argument of using the Hölder inequality. From Corollary 3.4 and Proposition 3.1, we obtain that, for  $q > \frac{2n}{n-1}$ ,  $\frac{n+1}{q} = \frac{n-1}{p'}$ , and  $f \in L_1$ ,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim R^{\alpha(R)} \|f\|_{L^p(S, d\sigma)},$$

where

$$\alpha(R) = \begin{cases} -\frac{n-1}{2} [1 - \frac{2n}{q(n-1)}], & \text{for } R \geq 2, \\ \frac{n-1}{q}, & \text{for } R \leq 1. \end{cases}$$

By scaling, when  $f \in L_M$  with  $M \in 2^{\mathbf{Z}}$ , under the condition  $\frac{n+1}{q} = \frac{n-1}{p'}$ ,

$$\|(fd\sigma)^\vee\|_{L_{t,x}^q(\mathbf{R} \times A_R)} \lesssim (RM)^{\alpha(RM)} \|f\|_{L^p(S, d\sigma)}.$$

Then for general  $f$ , we decompose it as follows,

$$f = \sum_{M: \text{dyadic}} f 1_{\{(\tau, \xi): \tau = |\xi|, M \leq |\xi| \leq 2M\}} = \sum_M f_M,$$

where  $f_M := f1_{\{(\tau, \xi): \tau=|\xi|, M \leq |\xi| \leq 2M\}}$ . Hence

$$\begin{aligned} \|(f d\sigma)^\vee\|_{L^q_{t,x}(\mathbf{R} \times \mathbf{R}^{n-1})} &= \left( \sum_R \|(f d\sigma)^\vee\|_{L^q_{t,x}(\mathbf{R} \times A_R)}^q \right)^{1/q} \\ &= \left( \sum_R \left\| \sum_M (f_M d\sigma)^\vee \right\|_{L^q_{t,x}(\mathbf{R} \times A_R)}^q \right)^{1/q} \lesssim \left( \sum_R \left( \sum_M \|(f_M d\sigma)^\vee\|_{L^q_{t,x}(\mathbf{R} \times A_R)}^q \right)^q \right)^{1/q} \\ &\lesssim \left( \sum_R \left( \sum_M (RM)^{\alpha(RM)} \|f_M\|_{L^p(S, d\sigma)} \right)^q \right)^{1/q} \lesssim \left( \sum_M \|f_M\|_{L^p(S, d\sigma)}^p \right)^{1/p} \sim \|f\|_{L^p(S, d\sigma)}, \end{aligned}$$

where  $R > 0$  and  $M > 0$  are dyadic numbers; for the last line, we have used the Schur's test since  $q > \frac{2n}{n-1} > p \geq 1$  and

$$\sup_{R>0} \sum_M (RM)^{\alpha(RM)} < \infty \text{ and } \sup_{M>0} \sum_R (RM)^{\alpha(RM)} < \infty.$$

Hence Theorem 1.1 follows.  $\square$

#### 4. SECOND PROOF OF THEOREM 1.1

To begin with the second proof, we introduce the following strengthening version of the Hausdorff-Young inequality [7, Chapter 4, Corollary 3.16].

**Lemma 4.1.** *If  $f \in L^p(\mathbf{R}^n)$ ,  $1 < p \leq 2$ , then  $\hat{f}$  belongs to  $L^{p,p'}$  and*

$$\|\hat{f}\|_{L^{p,p'}} \lesssim \|f\|_{L^p},$$

*or in its dual form, for any  $f \in L^{p,p'}$ ,*

$$\|\hat{f}\|_{L^{p,p'}} \lesssim \|f\|_{L^{p,p'}},$$

*where  $L^{p,q}$  denotes the Lorentz space for  $0 < p < \infty$ ,  $0 < q \leq \infty$ , which is defined via the equivalence that  $f \in L^{p,q}$  if and only if the norm  $\|f\|_{L^{p,q}} := \left( \frac{q}{p} \int_0^\infty (\lambda |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|^{1/p})^q \frac{d\lambda}{\lambda} \right)^{1/q}$  is finite with the usual modification weak- $L^p$  when  $q = \infty$ .*

We also introduce the following Hölder inequality in the Lorentz spaces [2, Chapter 5, Theorem 5.3.1].

**Lemma 4.2.** *If  $0 < p_1, p_2, p < \infty$  and  $0 < q_1, q_2, q \leq \infty$  obey  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$  and  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ , then*

$$\|fg\|_{L^{p,q}} \lesssim_{p_1, p_2, q_1, q_2} \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}},$$

*whenever the right-hand side norms are finite.*

Next we will present the second proof of Theorem 1.1, which is inspired by Nicola's short proof in [3] that the restriction conjecture for the sphere in  $\mathbf{R}^n$  implies that for cone in  $\mathbf{R} \times \mathbf{R}^n$ .

*The second proof of Theorem 1.1.* As in the first proof, it is sufficient to consider  $q > \frac{2n}{n-1}$  and  $\frac{n+1}{q} = \frac{n-1}{p'}$ . By changing to polar coordinate,

$$(f d\sigma)^\vee(t, x) = c_n r^{-\frac{n-2}{2}} \int_0^\infty e^{its} F(s) s^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(sr) ds.$$

Then by using Lemma 4.1 and exchanging the norms, we have

$$\begin{aligned} \|(fd\sigma)^\vee\|_{L_{t,x}^q} &\lesssim \left\| \left\| r^{-\frac{n-2}{2}+\frac{n-1}{q}} F(s) s^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(sr) \right\|_{L_s^{q',q}(0,\infty)} \right\|_{L_r^q(0,\infty)} \\ &\lesssim \left\| F(s) s^{\frac{n}{2}-1} \left\| r^{-\frac{n-2}{2}+\frac{n-1}{q}} J_{\frac{n}{2}-1}(rs) \right\|_{L_r^q(0,\infty)} \right\|_{L_s^{q',q}(0,\infty)}. \end{aligned}$$

We observe that for each  $s > 0$ , the integrand is bounded by

$$(4.1) \quad \left\| r^{-\frac{n-2}{2}+\frac{n-1}{q}} J_{\frac{n}{2}-1}(rs) \right\|_{L_r^q(1/s,\infty)} + \left\| r^{-\frac{n-2}{2}+\frac{n-1}{q}} J_{\frac{n}{2}-1}(rs) \right\|_{L_r^q(0,1/s)}.$$

On the one hand, from the definition of the Bessel function

$$J_{\frac{n}{2}-1}(r) = \frac{(r/2)^{\frac{n-2}{2}}}{\Gamma((n-1)/2)\Gamma(1/2)} \int_{-1}^1 e^{irs} (1-s^2)^{\frac{n-3}{2}} ds,$$

we obtain

$$J_{\frac{n}{2}-1}(r) \lesssim r^{\frac{n-2}{2}} \text{ for } n \geq 2 \text{ and } r \leq 1.$$

On the other hand, from the complete expansion of  $J_m$  when  $m = \frac{n}{2} - 1$  and the bound on  $E(r)$  in the proof of Proposition 3.3, we have

$$|J_{\frac{n}{2}-1}(r)| \lesssim r^{-1/2} + c_n r^{\frac{n}{2}-1} r^{-\frac{n+1}{2}} \lesssim r^{-1/2}, \text{ for } n \geq 2 \text{ and } r \geq 1.$$

Hence combining these two estimates on  $J_{\frac{n}{2}-1}$ , we obtain

$$(4.1) \lesssim s^{\frac{n}{2}-\frac{n}{q}-1}, \text{ if } q > \frac{2n}{n-1}.$$

Then by the fact that  $q > p$  and Lemma 4.2,

$$\begin{aligned} \|(fd\sigma)^\vee\|_{L_{t,x}^q} &\lesssim \|F(s) s^{\frac{n-2}{p}} s^{-\frac{n-2}{p}+n-\frac{n}{q}-2}\|_{L_s^{q',q}} \\ &\lesssim \|F(s) s^{\frac{n-2}{p}} s^{-\frac{n-2}{p}+n-\frac{n}{q}-2}\|_{L_s^{q',p}} \\ &\lesssim \|F(s) s^{\frac{n-2}{p}}\|_{L^{p,p}} \|s^{-\frac{n-2}{p}+n-\frac{n}{q}-2}\|_{L^{\frac{1}{1/q'-1/p},\infty}}. \end{aligned}$$

Note the condition  $\frac{n+1}{q} = \frac{n-1}{p'}$  implies that  $-\frac{n-2}{p} + n - \frac{n}{q} - 2 = -(\frac{1}{q'} - \frac{1}{p})$ . Hence

$$\|s^{-\frac{n-2}{p}+n-\frac{n}{q}-2}\|_{L^{\frac{1}{1/q'-1/p},\infty}} < \infty.$$

Therefore, by the fact that  $\|F(s) s^{\frac{n-2}{p}}\|_{L^{p,p}} = \|f\|_{L^p(S,d\sigma)}$ , we see that Theorem 1.1 follows.  $\square$

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